

## Complex Numbers and Quadratic Equations

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- A number of the form  $a + ib$ , where  $a$  and  $b$  are real numbers and  $i = \sqrt{-1}$ , is defined as a complex number.
- For the complex numbers  $z = a + ib$ ,  $a$  is called the real part (denoted by  $\text{Re } z$ ) and  $b$  is called the imaginary part (denoted by  $\text{Im } z$ ) of the complex number  $z$ .

**Example:** For the complex number  $z = \frac{-5}{9} + i\frac{\sqrt{3}}{17}$ ,  $\text{Re } z = \frac{-5}{9}$  and  $\text{Im } z = \frac{\sqrt{3}}{17}$

- Two complex numbers  $z_1 = a + ib$  and  $z_2 = c + id$  are equal if  $a = c$  and  $b = d$ .
- **Addition of complex numbers**

Two complex numbers  $z_1 = a + ib$  and  $z_2 = c + id$  can be added as,  
 $z_1 + z_2 = (a + ib) + (c + id) = (a + c) + i(b + d)$

- **Properties of addition of complex numbers:**
  - **Closure Law:** Sum of two complex numbers is a complex number. In fact, for two complex numbers  $z_1$  and  $z_2$ , such that  $z_1 = a + ib$  and  $z_2 = c + id$ , we obtain  $z_1 + z_2 = (a + c) + i(b + d)$ .
  - **Commutative Law:** For two complex numbers  $z_1$  and  $z_2$ ,

$$z_1 + z_2 = z_2 + z_1.$$

- **Associative Law:** For any three complex numbers  $z_1$ ,  $z_2$  and  $z_3$ ,

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3).$$

- **Existence of Additive Identity:** There exists a complex number  $0 + i0$  (denoted by  $0$ ), called the additive identity or zero complex number, such that for every complex number  $z$ ,  $z + 0 = z$ .
- **Existence of Additive Inverse:** For every complex number  $z = a + ib$ , there exists a complex number  $-a + i(-b)$  [denoted by  $-z$ ], called the additive inverse or negative of  $z$ , such that  $z + (-z) = 0$ .

- **Subtraction of complex numbers**

Given any two complex numbers  $z_1$ , and  $z_2$ , the difference  $z_1 - z_2$  is defined as  
 $z_1 - z_2 = z_1 + (-z_2)$ .

- **Multiplication of complex numbers:**



Two complex numbers  $z_1 = a + ib$  and  $z_2 = c + id$  can be multiplied as,  
 $z_1 z_2 = (ac - bd) + i(ad + bc)$

- Properties of multiplication of complex numbers:
- **Closure law:** The product of two complex numbers is a complex number.

In fact, for two complex numbers  $z_1$  and  $z_2$ , such that  $z_1 = a + ib$  and  $z_2 = c + id$ , we obtain  $z_1 z_2 = (ac - bd) + i(ad + bc)$ .

- **Commutative Law:** For any two complex numbers  $z_1$  and  $z_2$ ,  $z_1 z_2 = z_2 z_1$ .
- **Associative Law:** For any three complex numbers  $z_1$ ,  $z_2$  and  $z_3$ ,

$$(z_1 z_2) z_3 = z_1 (z_2 z_3).$$

- **Existence of Multiplicative Identity:** There exist a complex number  $1 + i0$  (denoted as 1), called the multiplicative identity, such that for every complex numbers  $z$ ,  $z \cdot 1 = z$ .
- **Existence of Multiplicative Inverse:** For every non-zero complex number  $z = a + ib$  ( $a \neq 0, b \neq 0$ ), we have the complex number

$$\frac{a}{a^2 + b^2} + i \frac{-b}{a^2 + b^2} \quad \left( \text{denoted by } \frac{1}{z} \text{ or } z^{-1} \right), \text{ called the multiplicative inverse of } z,$$

such that  $z \cdot \frac{1}{z} = 1$ .

**Example:** The multiplicative inverse of the complex number  $z = 2 - 3i$  can be found as,

$$z^{-1} = \frac{2}{(2)^2 + (-3)^2} + i \frac{(-3)}{(2)^2 + (-3)^2} = \frac{2}{13} - \frac{3}{13}i$$

- **Distributive Law:** For any three complex numbers  $z_1$ ,  $z_2$  and  $z_3$ ,
  - $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$
  - $(z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3$

- **Division of Complex Numbers**

Given any two complex number  $z_1$  and  $z_2$ , where  $z_2 \neq 0$ , the quotient  $\frac{z_1}{z_2}$  is defined as  $\frac{z_1}{z_2} = z_1 \frac{1}{z_2}$

**Example:** For  $z_1 = 1 + i$  and  $z_2 = 2 - 3i$ , the quotient  $\frac{z_1}{z_2}$  can be found as,

$$\begin{aligned}
\frac{z_1}{z_2} &= \frac{1+i}{2-3i} \\
&= (1+i) \left( \frac{1}{2-3i} \right) \\
&= (1+i) \left( \frac{2}{(2)^2+(-3)^2} + i \frac{(-3)}{(2)^2+(-3)^2} \right) \\
&= (1+i) \left( \frac{2}{13} - \frac{3}{13}i \right) \\
&= \left[ 1 \times \frac{2}{13} - 1 \times \left( -\frac{3}{13} \right) \right] + i \left[ 1 \times \left( -\frac{3}{13} \right) + 1 \times \frac{2}{13} \right] \\
&= \frac{5}{13} - \frac{1}{13}i
\end{aligned}$$

- **Property of Complex Numbers**

- For any integer  $k$ ,  $i^{4k} = 1$ ,  $i^{4k+1} = i$ ,  $i^{4k+2} = -1$ ,  $i^{4k+3} = -i$ .
- If  $a$  and  $b$  are negative real numbers, then

- **Modulus of a Complex Number**

The modulus of a complex number  $z = a + ib$ , is denoted by  $|z|$ , and is defined as the non-negative real number  $\sqrt{a^2 + b^2}$ , i.e.,  $|z| = \sqrt{a^2 + b^2}$ .

**Example 1:** If  $z = 2 - 3i$ , then  $|z| = \sqrt{(2)^2 + (-3)^2} = \sqrt{13}$

- **Conjugate of Complex Number**

The conjugate of a complex number  $z = a + ib$ , is denoted by  $\bar{z}$ , and is defined as the complex number  $a - ib$ , i.e.,  $\bar{z} = a - ib$ .

**Example 2:** Find the conjugate of  $\frac{2}{3+5i}$ .

**Solution:** We have

$$\begin{aligned}
& \frac{2}{3+5i} \\
&= \frac{2}{3+5i} \times \frac{3-5i}{3-5i} \\
&= \frac{2(3-5i)}{(3)^2 - (5i)^2} \\
&= \frac{2(3-5i)}{9-25i^2} \\
&= \frac{6-10i}{9+25} \quad \left( \because i^2 = -1 \right) \\
&= \frac{6-10i}{34} \\
&= \frac{6}{34} - \frac{10i}{34} \\
&= \frac{3}{17} - \frac{5i}{17}
\end{aligned}$$

Thus, the conjugate of  $\frac{2}{3+5i}$  is  $\frac{3}{17} + \frac{5i}{17}$ .

- Properties of modulus and conjugate of complex numbers:

For any three complex numbers  $z, z_1, z_2$ ,

- $z^{-1} = \frac{\bar{z}}{|z|^2}$  or  $z \cdot \bar{z} = |z|^2$ 
  - $|z_1 z_2| = |z_1| |z_2|$
  - $\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$ , provided  $|z_2| \neq 0$
  - $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$
  - $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$

- **Solutions of the quadratic equation when  $D < 0$ .**

The solutions of the quadratic equation  $ax^2 + bx + c = 0$ , where  $a, b, c \in \mathbb{R}$ ,  $a \neq 0$  are given

by  $x = \frac{-b \pm \sqrt{D}}{2a}$ , where  $D = b^2 - 4ac < 0$ .

**Example:** Solve  $2x^2 + 3ix + 2 = 0$

**Solution:** Here  $a = 2$ ,  $b = 3i$  and  $c = 2$

$$D = b^2 - 4ac$$

$$= (3i)^2 - 4 \times 2 \times 2$$

$$= -9 - 16$$

$$= -25 < 0$$

$$\therefore x = \frac{-b \pm \sqrt{D}}{2a}$$

$$\Rightarrow x = \frac{-3i \pm \sqrt{-25}}{2 \times 2}$$

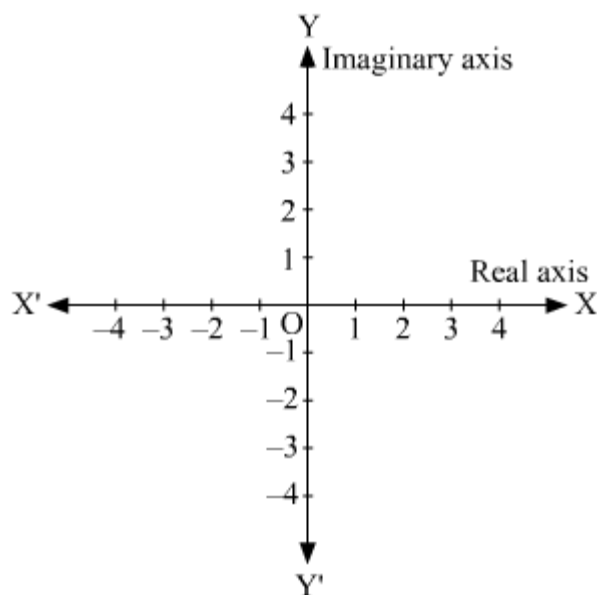
$$\Rightarrow x = \frac{-3i \pm 5i}{4}$$

$$\Rightarrow x = \frac{-3i+5i}{4} \text{ or } x = \frac{-3i-5i}{4}$$

$$\Rightarrow x = \frac{2i}{4} \text{ or } x = -\frac{8i}{4}$$

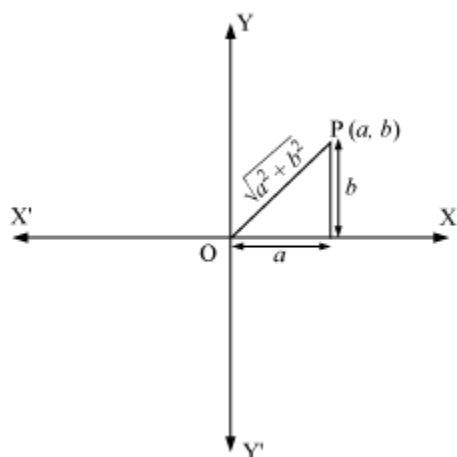
$$\Rightarrow x = \frac{i}{2} \text{ or } x = -2i$$

- **Argand plan:** Each complex number represents a unique point on Argand plane. An Argand plane is shown in the following figure.



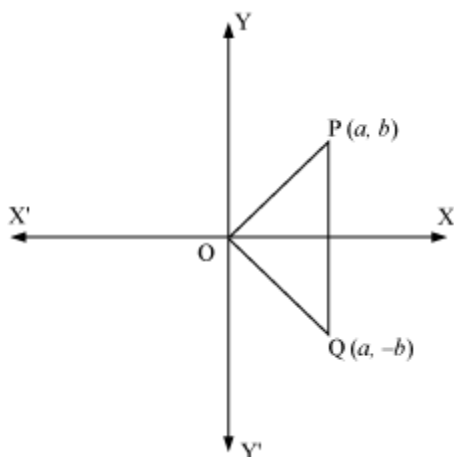
Here, x-axis is known as the **real axis** and y-axis is known as the **imaginary axis**.

- **Complex Number on Argand plane:** The complex number  $z = a + ib$  can be represented on an Argand plane as



In this figure,  $OP = \sqrt{a^2 + b^2} = |z|$

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- Thus, the modulus of a complex number  $z = a + ib$  is the distance between the point  $P(x, y)$  and the origin  $O$ .
- **Conjugate of Complex Number on Argand plane:**
- The conjugate of a complex number  $z = a + ib$  is  $\bar{z} = a - ib$ ,  $z$  and  $\bar{z}$  can be represented by the points  $P(a, b)$  and  $Q(a, -b)$  on the Argand plane as



Thus, on the Argand plane, the conjugate of a complex number is the mirror image of the complex number with respect to the real axis.

### • Polar representation of Complex Numbers

The polar form of the complex number  $z = x + iy$ ,  
is  $r(\cos \theta + i \sin \theta)$  where  $r = \sqrt{x^2 + y^2}$  (modulus of  $z$ )

and  $\cos \theta = \frac{x}{r}$ ,  $\sin \theta = \frac{y}{r}$  ( $\theta$  is known as the argument of  $z$ ).

The value of  $\theta$  is such that  $-\pi < \theta \leq \pi$ , which is called the principle argument of  $z$ .

**Example 2:** Represent the complex number  $z = \sqrt{2} - i\sqrt{2}$  in polar form.

**Solution:**  $z = \sqrt{2} - i\sqrt{2}$

Let  $\sqrt{2} = r \cos \theta$  and  $-\sqrt{2} = r \sin \theta$

By squaring and adding them, we have

$$2 + 2 = r^2 (\cos^2 \theta + \sin^2 \theta)$$

$$\Rightarrow r^2 = 4$$

$$\Rightarrow r = \sqrt{4} = 2$$

Thus,

$$\cos \theta = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$\sin \theta = \frac{-\sqrt{2}}{2} = \frac{-1}{\sqrt{2}} = \sin \left( 2\pi - \frac{\pi}{4} \right)$$

$$\Rightarrow \theta = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$$

Thus, the required polar form is  $2 \left( \cos \frac{7\pi}{4} + \sin \frac{7\pi}{4} \right)$ .